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Metallic ferromagnetism in the Hubbard model on three coupled chains

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Abstract. We study the Hubbard model on three coupled chains which has the saturated ferromagnetic ground states in a finite range of the electron filling factor ν. In the case that both the single-electron band gap between the highest and lower two bands and the on-site Coulomb interaction are infinitely large, we will rigorously show that the present model exhibits ferromagnetism for $\nu_0 < \nu < \frac{1}{3}$, where $0 \leqslant \nu_0 \leqslant \frac{1}{6}$ depending on the band structure. We will also give numerical results, which indicate that, even in the case that the band gap and the Coulomb interaction are not so large, the present ferromagnetism will survive for suitable electron filling factors.

1. Introduction

In spite of much research, mechanisms of ferromagnetism in itinerant electron systems are still unclear. It is widely believed that the spin-independent Coulomb interaction and the Pauli exclusion principle can cause ferromagnetism in itinerant electron systems, and the Hubbard model [1–6] is regarded as a simplified suitable model for the study of ferromagnetism. Even in this simplified model, however, we do not yet completely know under what conditions the ferromagnetic ground states are stable. A mean-field theory for the Hubbard model yields the so-called Stoner's criterion, which tells us that none too small values of the Coulomb interaction and the single-particle density of states at the Fermi surface are necessary for ferromagnetism in this model. The criterion predicts that the Hubbard model exhibits ferromagnetism in a wide range of parameters. Improved approximations with the inclusion of correlation effects explain some experimental facts [7]. However, this type of the approximation usually overestimates the tendency of ferromagnetism, so different approaches seem to be necessary to obtain a deep understanding of the origin of ferromagnetism in the itinerant electron systems.

One of the rigorous examples of ferromagnetism in the Hubbard model was given by Nagaoka [8], and independently by Thouless [9]. It was proved that the Hubbard model with an infinitely large Coulomb interaction has the saturated ferromagnetic ground states when there is precisely one hole. The theorem was extended by Tasaki to a general class of Hubbard models which satisfy a certain connectivity condition [10].

In the last decade, several rigorous examples of Hubbard models which exhibit ferrimagnetism or ferromagnetism were proposed by Lieb, Mielke, and Tasaki [11–14, 16, 17]. Lieb [14] showed that Hubbard models on the bipartite lattice at half-filling have the ferrimagnetic ground states for all positive values of the Coulomb interaction when the difference in the number of sites between the two sublattices is proportional to the number of the whole lattice sites. Later, Shen *et al* [15] showed the existence of ferrimagnetic

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long-range order in Lieb's model. Mielke and Tasaki [16] presented Hubbard models, whose single-electron Schrödinger equations have highly degenerate ground states, exhibiting ferromagnetism for all positive values of the Coulomb interaction. Tasaki [17] also presented non-singular Hubbard models which have dispersive single-electron bands and exhibit ferromagnetism in their ground states for sufficiently large finite values of the Coulomb interaction.

Despite this remarkable progress, little is known about metallic ferromagnetism. Lieb's ferrimagnetism and Tasaki's non-singular ferromagnetism are proved for a peculiar density of electrons. As for Mielke and Tasaki's flat-band ferromagnetism, it is proved that the models exhibit ferromagnetism when the number of electrons is not more than and sufficiently close to that of the degeneracy of single-electron ground states [16]. Due to the special feature of single-electron ground states, the charge gap of their models is always zero for the number of electrons less than that of single-electron ground states. As Mielke and Tasaki [16] said, however, a more detailed analysis seems necessary to determine whether their models are metallic or not.

In the Nagaoka–Thouless ferromagnetism there is exactly one hole and its motion maintains ferromagnetism. We also studied the Hubbard model which exhibits ferromagnetism due to one-electron motion [18]. But it is hardly expected that electric current in a bulk system can be produced by only one hole or one electron. There are many investigations to extend the Nagaoka–Thouless theorem to systems containing many holes [11, 19]. As far as we know, however, no conclusive result has been obtained in this direction at the present time.

Apart from simple one-orbital Hubbard models, there are some rigorous results about metallic ferromagnetism in certain Hubbard models with other types of interactions and in two-orbital (two-band) Hubbard models on a one-dimensional lattice [13, 20, 21]. The results rely strongly on the nature of one-dimensional systems and are obtained by applications of the Perron–Frobenius theorem.

Recently, Kohno [22] discussed the magnetic properties in the Hubbard model on a two-leg ladder. By using analytic and numerical calculations, he pointed out that this model exhibits metallic ferromagnetism for the electron filling factors from $\frac{1}{4}$ to $\frac{1}{2}$. In this model, the Coulomb interaction between electrons in the upper and the lower bands plays an important role in generating ferromagnetism.

In this paper, we will consider the Hubbard model on three coupled chains which exhibits ferromagnetism in a finite range of the electron filling factor ν. In the limit that both the single-electron band gap between the highest and the other two bands as well as the Coulomb interaction are infinitely large, we can rigorously show that the model has the saturated ferromagnetic ground states for $\nu_0 < \nu < \frac{1}{3}$, where ν_0 takes a value in the range $0 \leqslant \nu_0 \leqslant \frac{1}{6}$ depending on the structure of the lower two bands. Our result is obtained through the Perron–Frobenius argument as well as other examples of metallic ferromagnetism in one-dimensional systems.

In the infinite band-gap limit, a many-electron state with finite energy is constructed by using single-electron states in the lower two bands. We will see that the effective Hamiltonian of our model for the finite-energy states becomes a two-orbital Hubbard model with some interaction terms including ferromagnetic ones between two electrons in different orbitals. In the limit of the infinitely large Coulomb interaction, these ferromagnetic coupling terms become dominant, therefore, the present model exhibits ferromagnetism. It is noted that ferromagnetism in the present model is generated by the combination of the large band gap, the large on-site Coulomb interaction, the electron hopping, and the suitable electron filling.

In section 2 we will give a definition of the present model and observe single electron properties. In section 3 we will discuss the effective Hamiltonian and give some numerical results. In section 4 we will prove that the present model has the saturated ferromagnetic

ground states in the limit that the band gap and the Coulomb interaction are infinitely large.

2. Definition and single-electron problems

Let $\mathrm{L} = \{1, 2, \ldots, N\}$, where N is an arbitrary integer, and let Λ be the set defined by

$$\Lambda = \{i = (x, m) | x \in \mathrm{L}, m = 1, 2, 3\}. \tag{1}$$

With the hopping matrix elements defined below, we can regard Λ as three coupled chains with $3N$ sites. We impose open boundary conditions. We denote the number of elements in a set X by $|\mathrm{X}|$. As usual, we denote the creation, the annihilation, and the number operators for an electron on a site i with a spin $\sigma = \uparrow, \downarrow$ by $c^{\dagger}_{i,\sigma}$, $c_{i,\sigma}$, and $n_{i,\sigma}$, respectively. The creation and the annihilation operators satisfy the anticommutation relations

$$\{c_{i,\sigma}, c^{\dagger}_{j,\tau}\} = \delta_{ij}\delta_{\sigma\tau} \tag{2}$$

and

$$\{c_{i,\sigma}, c_{j,\tau}\} = \{c^{\dagger}_{i,\sigma}, c^{\dagger}_{j,\tau}\} = 0 \tag{3}$$

for any $i, j \in \Lambda$ and $\sigma, \tau = \uparrow, \downarrow$. By N_{e}, we denote the number of electrons. By ν, we denote the electron filling factor $N_{\mathrm{e}}/(2|\Lambda|)$. For $\alpha = 1, 2, 3$, we will define local spin operators by $S^{(\alpha)}_i = \sum_{\sigma,\tau=\uparrow,\downarrow} c^{\dagger}_{i,\sigma}(p^{(\alpha)}_{\sigma\tau}/2)c_{i,\tau}$, where $p^{(\alpha)}$ are the Pauli matrices. We will also define the total spin operator by $S^{(\alpha)}_{\mathrm{tot}} = \sum_{i\in\Lambda} S^{(\alpha)}_i$, and denote the eigenvalues of $(\boldsymbol{S}_{\mathrm{tot}})^2 = \sum_{\alpha=1}^{3}(S^{(\alpha)}_{\mathrm{tot}})^2$ and $S^{(3)}_{\mathrm{tot}}$ as $S_{\mathrm{tot}}(S_{\mathrm{tot}} + 1)$ and M, respectively.

We will consider the Hubbard model on the lattice Λ to be defined by

$$\mathcal{H}(\Lambda) = \mathcal{H}_{\mathrm{hop}}(\Lambda) + \mathcal{H}_{\mathrm{int}}(\Lambda) \tag{4}$$

with

$$\mathcal{H}_{\mathrm{hop}}(\Lambda) = \sum_{i,j\in\Lambda}\sum_{\sigma=\uparrow,\downarrow} t_{ij}c^{\dagger}_{i,\sigma}c_{j,\sigma} \tag{5}$$

$$\mathcal{H}_{\mathrm{int}}(\Lambda) = U\sum_{i\in\Lambda} n_{i,\uparrow}n_{i,\downarrow} \tag{6}$$

where the hopping matrix elements are symmetric and given by $t_{(x,1)(x+1,1)} = t_{(x,3)(x+1,3)} = -2t$, $t_{(x,2)(x+1,2)} = -4t$, $t_{(x,1)(x+1,2)} = t_{(x,2)(x+1,1)} = t_{(x,2)(x+1,3)} = t_{(x,3)(x+1,2)} = 2t$, $t_{(x,1)(x,2)} = t_{(x,2)(x,3)} = 2v_2 + v_3$, $t_{(x,1)(x,3)} = v_1 - v_2 + v_3$, $t_{(x,1)(x,1)} = t_{(x,3)(x,3)} = -v_1 - v_2 + v_3$, $t_{(x,2)(x,2)} = -4v_2 + v_3$ with $t > 0$, v_i being an arbitrary real number (figure 1). We define the new fermion operators $a_{[x,m],\sigma}$ for $x \in \mathrm{L}$, $m = 1, 2, 3$ and $\sigma = \uparrow, \downarrow$ by

$$a_{[x,1],\sigma} = \frac{1}{\sqrt{2}}(c_{(x,1),\sigma} - c_{(x,3),\sigma}) \tag{7}$$

$$a_{[x,2],\sigma} = \frac{1}{\sqrt{6}}(c_{(x,1),\sigma} - 2c_{(x,2),\sigma} + c_{(x,3),\sigma}) \tag{8}$$

$$a_{[x,3],\sigma} = \frac{1}{\sqrt{3}}(c_{(x,1),\sigma} + c_{(x,2),\sigma} + c_{(x,3),\sigma}) \tag{9}$$

which satisfy the anticommutation relations

$$\{a_{[x,m],\sigma}, a^{\dagger}_{[y,m']\tau}\} = \delta_{xy}\delta_{mm'}\delta_{\sigma\tau} \tag{10}$$

and

$$\{a_{[x,m],\sigma}, a_{[y,m'],\tau}\} = \{a^{\dagger}_{[x,m],\sigma}, a^{\dagger}_{[y,m'],\tau}\} = 0 \tag{11}$$

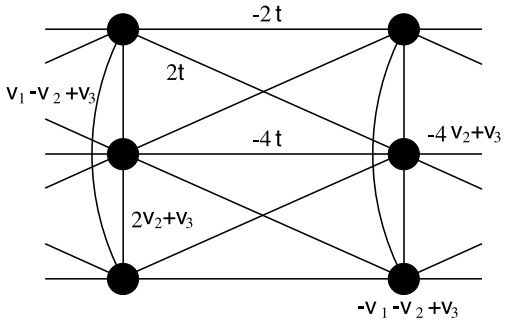

Figure 1. The hopping matrix elements.

for any $x, y \in \mathrm{L}$, $m, m' = 1, 2, 3$ and $\sigma, \tau = \uparrow, \downarrow$. By using these operators, the hopping part of the Hamiltonian can be rewritten as

$$\mathcal{H}_{\mathrm{hop}}(\Lambda) = \sum_{m=1,2} \mathcal{H}_{\mathrm{hop}}^{(t_m)}(\Lambda) + \sum_{m=1,2,3} \mathcal{H}_{\mathrm{hop}}^{(\tilde{v}_m)}(\Lambda) \tag{12}$$

where

$$\mathcal{H}_{\mathrm{hop}}^{(t_m)}(\Lambda) = -t_m \sum_{x \in \mathrm{L}\setminus\{N\}} \sum_{\sigma=\uparrow,\downarrow} (a_{[x,m],\sigma}^{\dagger} a_{[x+1,m],\sigma} + a_{[x+1,m],\sigma}^{\dagger} a_{[x,m],\sigma}) \tag{13}$$

$$\mathcal{H}_{\mathrm{hop}}^{(\tilde{v}_m)}(\Lambda) = \tilde{v}_m \sum_{x \in \mathrm{L}} \sum_{\sigma=\uparrow,\downarrow} a_{[x,m],\sigma}^{\dagger} a_{[x,m],\sigma} \tag{14}$$

with $t_1 = 2t$, $t_2 = 6t$, $\tilde{v}_1 = -2v_1$, $\tilde{v}_2 = -6v_2$, and $\tilde{v}_3 = 3v_3$.

Let us consider the single-electron Schrödinger equation. Let Φ_0 be a state with no electron on the lattice Λ. Since the states $\{a_{[x,m],\sigma}^{\dagger}\Phi_0\}$ are linearly independent and orthonormalized, a single-electron state can be expanded as

$$\Phi_{\mathrm{single}} = \sum_{x \in \mathrm{L}, m=1,2,3} u_{[x,m]} a_{[x,m],\sigma}^{\dagger} \Phi_0. \tag{15}$$

The single-electron equation $\mathcal{H}_{\mathrm{hop}}(\Lambda)\Phi_{\mathrm{single}} = \varepsilon\Phi_{\mathrm{single}}$ is reduced to the $N \times N$ matrix equations of the form

$$\mathbf{H}_{\mathrm{hop}}^{(m)} \boldsymbol{u}^{(m)} = \varepsilon^{(m)} \boldsymbol{u}^{(m)} \tag{16}$$

with $m = 1, 2, 3$, where $\boldsymbol{u}^{(m)}(x) = u_{[x,m]}$ and

$$[\mathbf{H}_{\mathrm{hop}}^{(m)}](x, y) = \langle a_{[x,m],\sigma}^{\dagger}\Phi_0, \mathcal{H}_{\mathrm{hop}}(\Lambda) a_{[y,m],\sigma}^{\dagger}\Phi_0 \rangle. \tag{17}$$

We denote the eigenvalue of $\mathbf{H}_{\mathrm{hop}}^{(m)}$ by $\varepsilon_l^{(m)}$, where $l = 1, \ldots, N$ and $\varepsilon_l^{(m)} \leqslant \varepsilon_{l'}^{(m)}$ if $l < l'$. We can easily find that $\mathbf{H}_{\mathrm{hop}}^{(3)}$ has the N-fold degenerate eigenvalue $\tilde{v}_3$. We can also easily see that v_m controls the gap between bands and t controls $\varepsilon_N^{(1)} - \varepsilon_1^{(1)}$ and $\varepsilon_N^{(2)} - \varepsilon_1^{(2)}$. Though we work on the lattice with open boundary conditions, it is useful to keep in mind the dispersion relations on one with periodic boundary conditions. We will show these in figure 2.

3. Effective Hamiltonian and discussion

3.1. Effective Hamiltonian

In this section, we will derive the effective interaction in the limit $v_3 \to \infty$ and see how ferromagnetic couplings are produced by Coulomb interaction terms. We will see that the

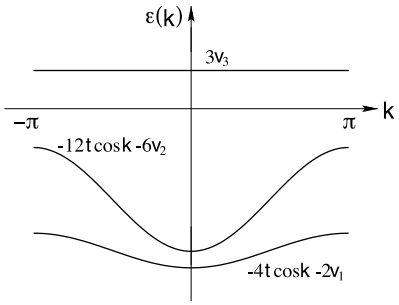

Figure 2. The dispersion relations.

effective Hamiltonian of our model in the limit $v_3, U \to \infty$ is the two-orbital Hubbard model in a certain limit, which was studied by Kubo [20], and Kusakabe and Aoki [21].

In the limit $v_3 \to \infty$, an electron can occupy a state in the lower bands. We will define the projection operator P by

$$P = \prod_{x\in \mathrm{L}} \prod_{\sigma=\uparrow,\downarrow} (1 - a^{\dagger}_{[x,3],\sigma} a_{[x,3],\sigma}) \tag{18}$$

which projects a state onto the subspace without occupancy of single-electron states with the energy $\tilde{v}_3$. For the fermion operator $a_{[x,m],\sigma}$, we introduce a number operator, $\tilde{n}_{[x,m],\sigma}$, and a spin operator, $\tilde{S}^{(\alpha)}_{[x,m]}$, by

$$\tilde{n}_{[x,m],\sigma} = a^{\dagger}_{[x,m],\sigma} a_{[x,m],\sigma} \tag{19}$$

$$\tilde{S}^{(\alpha)}_{[x,m]} = \sum_{\sigma,\tau=\uparrow,\downarrow} a^{\dagger}_{[x,m],\sigma} (p^{(\alpha)}_{\sigma\tau}/2) a_{[x,m],\tau}. \tag{20}$$

We also define $\tilde{n}_{[x,m]}$ by $\tilde{n}_{[x,m]} = \sum_{\sigma=\uparrow,\downarrow} \tilde{n}_{[x,m],\sigma}$. Using these operators, we find that the effective form of the Coulomb interaction term becomes

$$\begin{aligned} P\mathcal{H}_{\mathrm{int}}(\Lambda)P = \sum_{x\in \mathrm{L}} \Bigg(& \frac{U}{3} \left(\frac{\tilde{n}_{[x,1]}\tilde{n}_{[x,2]}}{4} - \tilde{\boldsymbol{S}}_{[x,1]} \cdot \tilde{\boldsymbol{S}}_{[x,2]} \right) + \frac{4U}{9} \tilde{n}_{[x,2],\uparrow} \tilde{n}_{[x,2],\downarrow} \\ & + \frac{U}{2} \left(a^{\dagger}_{[x,1],\uparrow} a^{\dagger}_{[x,1],\downarrow} + \frac{1}{3} a^{\dagger}_{[x,2],\uparrow} a^{\dagger}_{[x,2],\downarrow} \right) \\ & \times \left(a_{[x,1],\downarrow} a_{[x,1],\uparrow} + \frac{1}{3} a_{[x,2],\downarrow} a_{[x,2],\uparrow} \right) \Bigg) P. \end{aligned} \tag{21}$$

The first term in the right-hand side of equation (21) is a ferromagnetic spin Hamiltonian. The large band gap effectively changes the on-site Coulomb interaction into the ferromagnetic interaction between electrons represented by $a^{\dagger}_{[x,1],\sigma}$ and $a^{\dagger}_{[x,2],\tau}$.

Each term in the right-hand side of equation (21) is positive semidefinite. Therefore, in the limit $U \to \infty$, any finite-energy state Φ, in which no single-electron state with the energy $\tilde{v}_3$ is occupied, must satisfy

$$\left\langle \Phi, \left(\frac{\tilde{n}_{[x,1]}\tilde{n}_{[x,2]}}{4} - \tilde{\boldsymbol{S}}_{[x,1]} \cdot \tilde{\boldsymbol{S}}_{[x,2]} \right) \Phi \right\rangle = 0 \tag{22}$$

$$a_{[x,1],\downarrow} a_{[x,1],\uparrow} \Phi = 0 \tag{23}$$

$$a_{[x,2],\downarrow} a_{[x,2],\uparrow} \Phi = 0. \tag{24}$$

Equation (22) is satisfied if two electrons represented by $a^{\dagger}_{[x,1],\sigma}$ and $a^{\dagger}_{[x,2],\tau}$ are coupled ferromagnetically. These conditions imply that locally allowed states on a sublattice $\Lambda_x = \{i = (x, m) | m = 1, 2, 3\}$ are no-electron, single-electron, and triplet two-electron states. Now

we will regard $x \in \mathrm{L}$ as a site and m as an orbital number. The hopping term $\mathcal{H}_{\mathrm{hop}}(\Lambda)$ without $\mathcal{H}_{\mathrm{hop}}^{\tilde{v}_3}(\Lambda)$ is regarded to represent the nearest-neighbour hopping of the electron in the same orbital and the potential energy of the orbitals. Therefore, in the limit $v_3, U \to \infty$, our model is reduced to the two-orbital Hubbard model where double-occupancy of an orbital and the doubly occupied singlet state on a site are forbidden. It was proved by Kubo [20] that such a model has the saturated ferromagnetic ground states for the electron filling factors from $\frac{1}{4}$ to $\frac{1}{2}$, which correspond to $\frac{1}{6} < \nu < \frac{1}{3}$ in the present model.

In section 4 we will give direct proof for the present model including an extension to other ranges of the electron filling factor where Kubo's proof is not applicable. Although the proof is mathematically similar to that by Kubo, it is emphasized that the mechanism of ferromagnetism is quite different. In the two-orbital Hubbard model treated by Kubo, the exclusion of the doubly occupied singlet state is due to the infinitely large inter-orbital interaction which favours the ferromagnetic coupling between electrons in different orbitals on the same site. In the present model, however, there is no explicit ferromagnetic coupling. The infinitely large ferromagnetic coupling is generated by the large band gap or equivalently the restriction of the single-electron states, and the large Coulomb interaction.

3.2. Ferromagnetism for finite values of v_3 and U

In this section, we will give some numerical results. We choose $v_1/t = 14.0$, $v_2/t = 2.0$ so that $\varepsilon_N^{(1)} < \varepsilon_1^{(2)}$ will hold. In this case, as we will see in section 4, the ground states of $\mathcal{H}(\Lambda)$ in the limit $v_3, U \to \infty$ have $S_{\mathrm{tot}} = N_{\mathrm{e}}/2$ for $\frac{1}{6} < \nu < \frac{1}{3}$. We have numerically calculated the ground state energy for finite values of v_3 and U, and estimated a level crossing point between the saturated ferromagnetic and other states within the $M = 0(\frac{1}{2})$ subspace for the even(odd) number of electrons. The calculations have been performed on lattices with open boundary conditions for $\frac{1}{6} < \nu < \frac{1}{3}$. In figure 3, we plotted the critical values of v_3 in the limit $U \to \infty$ on lattices with $9, 12$, and 15 sites as a function of the electron filling factor. On the lattice with nine sites we have numerically found that the ground state has $S_{\mathrm{tot}} = 0(\frac{1}{2})$ for even(odd) N_{e} below the critical point, which is expected on larger lattices. In order to obtain the critical value of U in the limit $v_3 \to \infty$, we have diagonalized numerically the effective Hamiltonian $P\mathcal{H}(\Lambda)P$ on lattices with $9, 12, 15$ and 18 sites and the results are shown in figure 4. The critical values of the Coulomb interaction for various values of v_3 are shown in figure 5. Although the lattice sizes in the present calculation are small to predict correct nature in the thermodynamic limit, it seems that the present ferromagnetism survives for suitable electron filling factors even when the values of v_3 and U are not so large.

3.3. Generalization

First we note that flatness of the single-electron band, characterized by the energy $\tilde{v}_3$, is not important. Let us consider the Hamiltonian $\mathcal{H}(\Lambda) + \mathcal{H}'(\Lambda)$, where $\mathcal{H}'(\Lambda)$ is defined by

$$\mathcal{H}'(\Lambda) = \tilde{t}_3 \sum_{x \in \mathrm{L}\setminus\{N\}} \sum_{\sigma=\uparrow,\downarrow} (a^\dagger_{[x,3],\sigma} \pm a^\dagger_{[x+1,3],\sigma})(a_{[x,3],\sigma} \pm a_{[x+1,3],\sigma}) \tag{25}$$

with $\tilde{t}_3 > 0$. The added Hamiltonian $\mathcal{H}'(\Lambda)$ changes the flat band into a dispersive one. Since $\mathcal{H}'(\Lambda)$ is a positive semidefinite operator, if the lowest-energy state of $\mathcal{H}(\Lambda)$ is a zero-energy state for the Hamiltonian $\mathcal{H}'(\Lambda)$, then that will also be the ground state of the whole Hamiltonian $\mathcal{H}(\Lambda) + \mathcal{H}'(\Lambda)$.

Provided that the electron filling factor is less than $\frac{1}{3}$ and the flat band is the highest, the lowest-energy state of $\mathcal{H}(\Lambda)$ will be a zero-energy state of $\mathcal{H}'(\Lambda)$, if it is the saturated

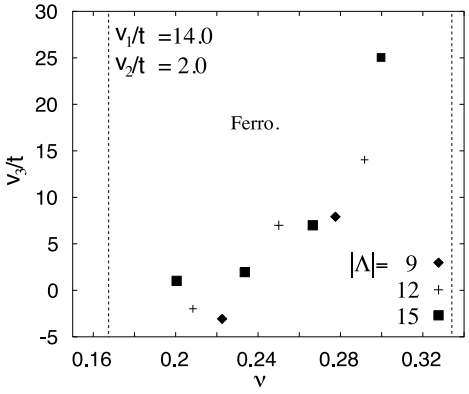

Figure 3. The critical values of v_3, above which the ground states of $\mathcal{H}(\Lambda)$ in the limit $U \to \infty$ have $S_{\rm tot} = N_{\rm e}/2$, as a function of filling factor ν.

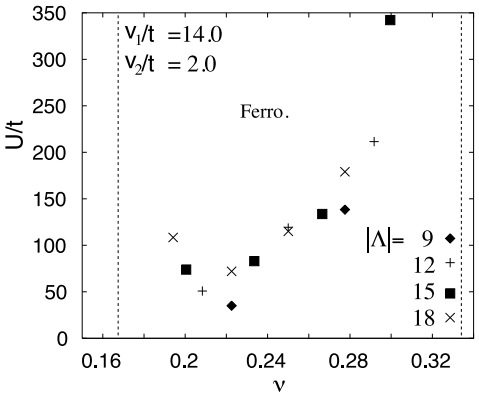

Figure 4. The critical values of U, above which the ground states of $\mathcal{H}(\Lambda)$ in the limit $v_3 \to \infty$ have $S_{\rm tot} = N_{\rm e}/2$, as a function of filling factor ν.

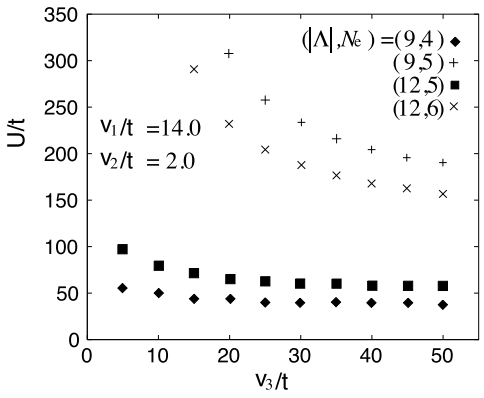

Figure 5. The band-gap v_3 dependence of the critical values of U, above which the ground states of $\mathcal{H}(\Lambda)$ have $S_{\rm tot} = N_{\rm e}/2$.

ferromagnetic state. Therefore, if $\mathcal{H}(\Lambda)$ has the saturated ferromagnetic ground states, then $\mathcal{H}(\Lambda) + \mathcal{H}'(\Lambda)$ does also.

Next we will briefly comment on models which can be treated by the same way. Let us introduce new fermion operators by

$$b_{[x,1],\sigma} = \frac{1}{\|b_{[x,1],\sigma}\|}(\lambda_x^{(3)} c_{(x,1),\sigma} - \lambda_x^{(1)} c_{(x,3),\sigma}) \tag{26}$$

$$b_{[x,2],\sigma} = \frac{1}{\|b_{[x,2],\sigma}\|}(\lambda_x^{(1)}\lambda_x^{(2)} c_{(x,1),\sigma} - ((\lambda_x^{(1)})^2 + (\lambda_x^{(3)})^2) c_{(x,2),\sigma} + \lambda_x^{(2)}\lambda_x^{(3)} c_{(x,3),\sigma}) \tag{27}$$

$$b_{[x,3],\sigma} = \frac{1}{\|b_{[x,3],\sigma}\|}(\lambda_x^{(1)} c_{(x,1),\sigma} + \lambda_x^{(2)} c_{(x,2),\sigma} + \lambda_x^{(3)} c_{(x,3),\sigma}) \tag{28}$$

$$\|b_{[x,1],\sigma}\| = ((\lambda_x^{(1)})^2 + (\lambda_x^{(3)})^2)^{1/2} \tag{29}$$

$$\|b_{[x,2],\sigma}\| = (((\lambda_x^{(1)})^2 + (\lambda_x^{(3)})^2)((\lambda_x^{(1)})^2 + (\lambda_x^{(2)})^2 + (\lambda_x^{(3)})^2))^{1/2} \tag{30}$$

$$\|b_{[x,3],\sigma}\| = ((\lambda_x^{(1)})^2 + (\lambda_x^{(2)})^2 + (\lambda_x^{(3)})^2)^{1/2} \tag{31}$$

for $x \in \mathrm{L}$ and $\sigma = \uparrow, \downarrow$. By using these new operators, we will define the model Hamiltonian on the lattice Λ by

$$\overline{\mathcal{H}}(\Lambda) = \sum_{m=1,2,3} (\overline{\mathcal{H}}_{\rm hop}^{(t_m)}(\Lambda) + \overline{\mathcal{H}}_{\rm hop}^{(v_m)}(\Lambda)) + \mathcal{H}_{\rm int}(\Lambda) \tag{32}$$

with

$$\overline{\mathcal{H}}_{\rm hop}^{(t_m)}(\Lambda) = -\sum_{x\in {\rm L}\setminus\{N\}}\sum_{\sigma=\uparrow,\downarrow} t_m(x,x+1)(b^{\dagger}_{[x,m],\sigma}b_{[x+1,m],\sigma} + b^{\dagger}_{[x+1,m],\sigma}b_{[x,m],\sigma}) \tag{33}$$

$$\overline{\mathcal{H}}_{\rm hop}^{(v_m)}(\Lambda) = \sum_{x\in {\rm L}}\sum_{\sigma=\uparrow,\downarrow} v_m(x)b^{\dagger}_{[x,m],\sigma}b_{[x,m],\sigma} \tag{34}$$

where $t_m(x,x+1) > 0$ for all $x \in {\rm L}\setminus\{N\}$ and $m = 1, 2$. We can also prove that the ground states of $\overline{\mathcal{H}}(\Lambda)$ have $S_{\rm tot} = N_{\rm e}/2$ in the limit of $v_3(x) \to \infty$ for all $x \in {\rm L}$ and $U \to \infty$.

4. Proof

4.1. Finite-energy states

The collection of states

$$\prod_{m=1,2,3}\prod_{\sigma=\uparrow,\downarrow}\prod_{x\in {\rm S}_{m,\sigma}} a^{\dagger}_{[x,m],\sigma}\Phi_0 \tag{35}$$

with arbitrary subsets ${\rm S}_{m,\sigma} \subset {\rm L}$ such that $\sum_{m=1,2,3}\sum_{\sigma=\uparrow,\downarrow}|{\rm S}_{m,\sigma}| = N_{\rm e}$ forms a complete orthonormalized basis of the $N_{\rm e}$-electron Hilbert space. We will expand a state by using this basis. In the limit $v_3, U \to \infty$, a finite-energy state Φ must satisfy

$$a_{[x,3],\sigma}\Phi = 0 \qquad \text{for all} \quad x \in {\rm L} \quad \text{and} \quad \sigma = \uparrow,\downarrow \tag{36}$$

$$c_{i,\downarrow}c_{i,\uparrow}\Phi = 0 \qquad \text{for all} \quad i \in \Lambda. \tag{37}$$

From condition (36), we have

$$\Phi = \sum_{\substack{{\rm S}_{1,\uparrow},{\rm S}_{1,\downarrow},{\rm S}_{2,\uparrow},{\rm S}_{2,\downarrow}\subset {\rm L}\\ |{\rm S}_{1,\uparrow}|+|{\rm S}_{1,\downarrow}|+|{\rm S}_{2,\uparrow}|+|{\rm S}_{2,\downarrow}|=N_{\rm e}}} g({\rm S}_{1,\uparrow},{\rm S}_{1,\downarrow},{\rm S}_{2,\uparrow},{\rm S}_{2,\downarrow})\Phi({\rm S}_{1,\uparrow},{\rm S}_{1,\downarrow},{\rm S}_{2,\uparrow},{\rm S}_{2,\downarrow}) \tag{38}$$

with

$$\Phi({\rm S}_{1,\uparrow},{\rm S}_{1,\downarrow},{\rm S}_{2,\uparrow},{\rm S}_{2,\downarrow}) = \prod_{m=1,2}\prod_{\sigma=\uparrow,\downarrow}\prod_{x\in {\rm S}_{m,\sigma}} a^{\dagger}_{[x,m],\sigma}\Phi_0. \tag{39}$$

Let us examine condition (37). We first choose $i = (y,2)$. Then, we obtain

$$g({\rm S}_{1,\uparrow},{\rm S}_{1,\downarrow},{\rm S}_{2,\uparrow},{\rm S}_{2,\downarrow}) = 0 \tag{40}$$

for ${\rm S}_{2,\uparrow}, {\rm S}_{2,\downarrow}$ such that $y \in {\rm S}_{2,\uparrow} \cap {\rm S}_{2,\downarrow}$. This holds for all $y \in {\rm L}$. There are at most three electrons on the sublattice $\Lambda_y = \{i = (y,m) | m = 1,2,3\}$, so for any y we can write a finite-energy state as

$$\Phi = \Psi_{y,0} + \Psi_{y,1} + \Psi_{y,2} + \Psi_{y,3} \tag{41}$$

where $\Psi_{y,k}$ is a state which contains k electrons on Λ_y. We easily see that $c_{(y,m),\downarrow}c_{(y,m),\uparrow}\Psi_{y,k} = 0$ if $k = 0, 1$. Since $c_{(y,m),\downarrow}c_{(y,m),\uparrow}\Psi_{y,2}$ and $c_{(y,m),\downarrow}c_{(y,m),\uparrow}\Psi_{y,3}$ are linearly independent, each must be zero. Taking into account condition (40), the state $\Psi_{y,3}$ can be written as

$$\Psi_{y,3} = a^{\dagger}_{[y,1],\uparrow}a^{\dagger}_{[y,1],\downarrow}a^{\dagger}_{[y,2],\uparrow}\tilde{\Psi}^{(1)}_{y,3} + a^{\dagger}_{[y,1],\uparrow}a^{\dagger}_{[y,1],\downarrow}a^{\dagger}_{[y,2],\downarrow}\tilde{\Psi}^{(2)}_{y,3} \tag{42}$$

where $\tilde{\Psi}^{(l)}_{y,3}$ is a state which contains no electron on Λ_y. Operating $c_{(y,1),\downarrow}c_{(y,1),\uparrow}$ on this, we have

$$\frac{1}{2}a^{\dagger}_{[y,2],\uparrow}\tilde{\Psi}^{(1)}_{y,3} - \frac{1}{2\sqrt{3}}a^{\dagger}_{[y,1],\uparrow}\tilde{\Psi}^{(1)}_{y,3} + \frac{1}{2}a^{\dagger}_{[y,2],\downarrow}\tilde{\Psi}^{(2)}_{y,3} - \frac{1}{2\sqrt{3}}a^{\dagger}_{[y,1],\downarrow}\tilde{\Psi}^{(2)}_{y,3} = 0 \tag{43}$$

which implies $\Psi_{y,3}$ should be zero. Similarly, we rewrite $\Psi_{y,2}$ as

$$\Psi_{y,2} = a^\dagger_{[y,1],\uparrow} a^\dagger_{[y,1],\downarrow} \tilde{\Psi}^{(1)}_{y,2} + a^\dagger_{[y,1],\uparrow} a^\dagger_{[y,2],\downarrow} \tilde{\Psi}^{(2)}_{y,2} + a^\dagger_{[y,1],\downarrow} a^\dagger_{[y,2],\uparrow} \tilde{\Psi}^{(3)}_{y,2} + a^\dagger_{[y,1],\uparrow} a^\dagger_{[y,2],\uparrow} \tilde{\Psi}^{(4)}_{y,2} + a^\dagger_{[y,1],\downarrow} a^\dagger_{[y,2],\downarrow} \tilde{\Psi}^{(5)}_{y,2} \tag{44}$$

where $\tilde{\Psi}^{(l)}_{y,2}$ is a state which contains no electron on Λ_y. Operating $c_{(y,1),\downarrow} c_{(y,1),\uparrow}$ and $c_{(y,3),\downarrow} c_{(y,3),\uparrow}$ on this, we obtain

$$\tilde{\Psi}^{(1)}_{y,2} = 0 \tag{45}$$

$$\tilde{\Psi}^{(2)}_{y,2} = \tilde{\Psi}^{(3)}_{y,2}. \tag{46}$$

Equation (45) is equivalent to

$$g(\mathrm{S}_{1,\uparrow}, \mathrm{S}_{1,\downarrow}, \mathrm{S}_{2,\uparrow}, \mathrm{S}_{2,\downarrow}) = 0 \tag{47}$$

for $\mathrm{S}_{1,\uparrow}, \mathrm{S}_{1,\downarrow}$ such that $y \in \mathrm{S}_{1,\uparrow} \cap \mathrm{S}_{1,\downarrow}$. From (40) and (47), we find that a finite-energy state can be expanded as

$$\Phi = \sum_{\substack{\mathrm{S}_1, \mathrm{S}_2 \subset \mathrm{L} \\ |\mathrm{S}_1| + |\mathrm{S}_2| = N_\mathrm{e}}} \sum_{([\sigma]_{\mathrm{S}_1}, [\tau]_{\mathrm{S}_2})} g(\mathrm{S}_1, \mathrm{S}_2, ([\sigma]_{\mathrm{S}_1}, [\tau]_{\mathrm{S}_2})) \Phi(\mathrm{S}_1, \mathrm{S}_2, ([\sigma]_{\mathrm{S}_1}, [\tau]_{\mathrm{S}_2})) \tag{48}$$

$$\Phi(\mathrm{S}_1, \mathrm{S}_2, ([\sigma]_{\mathrm{S}_1}, [\tau]_{\mathrm{S}_2})) = \prod_{x \in \mathrm{S}_1} a^\dagger_{[x,1],\sigma(x)} \prod_{x \in \mathrm{S}_2} a^\dagger_{[x,2],\tau(x)} \Phi_0 \tag{49}$$

where $([\sigma]_{\mathrm{S}_1}, [\tau]_{\mathrm{S}_2}) = (\sigma(x_1), \ldots, \sigma(x_{|\mathrm{S}_1|}), \tau(y_1), \ldots, \tau(y_{|\mathrm{S}_2|}))$ with $\sigma(x_l), \tau(y_l) = \uparrow, \downarrow$ and the summation $\sum_{([\sigma]_{\mathrm{S}_1}, [\tau]_{\mathrm{S}_2})}$ is taken over all these spin configurations. Now we require that the product in equation (49) is ordered so that $a^\dagger_{[x,m],\sigma}$ is always on the left of $a^\dagger_{[y,m],\tau}$ if $x < y$. From (46) or operating $c_{(y,1),\downarrow} c_{(y,1),\uparrow}$ on Φ in the form of equation (48) again, we obtain

$$g(\mathrm{S}_1, \mathrm{S}_2, ([\sigma]_{\mathrm{S}_1}, [\tau]_{\mathrm{S}_2})) = g(\mathrm{S}_1, \mathrm{S}_2, ([\tilde{\sigma}]_{\mathrm{S}_1} [\tilde{\tau}]_{\mathrm{S}_2})_y) \qquad \text{for all} \quad y \in \mathrm{S}_1 \cap \mathrm{S}_2 \tag{50}$$

where $([\tilde{\sigma}]_{\mathrm{S}_1}, [\tilde{\tau}]_{\mathrm{S}_2})_y$ is obtained from $([\sigma]_{\mathrm{S}_1}, [\tau]_{\mathrm{S}_2})$ through the relations $\tilde{\sigma}(x) = \tau(x)$ and $\tilde{\tau}(x) = \sigma(x)$ for $x = y$, and $\tilde{\sigma}(x) = \sigma(x)$ and $\tilde{\tau}(x) = \tau(x)$ for otherwise. In consequence, in the limit of infinitely large v_3 and U, a finite energy state can be expanded in the form of equation (48) with condition (50).

We can write equation (48) as

$$\Phi = \sum_{\substack{N_1, N_2 \\ N_1 + N_2 = N_\mathrm{e}}} \Phi_{(N_1, N_2)} \tag{51}$$

with

$$\Phi_{(N_1, N_2)} = \sum_{\substack{\mathrm{S}_1, \mathrm{S}_2 \subset \mathrm{L} \\ |\mathrm{S}_1| = N_1, |\mathrm{S}_2| = N_2}} \sum_{([\sigma]_{\mathrm{S}_1}, [\tau]_{\mathrm{S}_2})} g(\mathrm{S}_1, \mathrm{S}_2, ([\sigma]_{\mathrm{S}_1}, [\tau]_{\mathrm{S}_2})) \Phi(\mathrm{S}_1, \mathrm{S}_2, ([\sigma]_{\mathrm{S}_1}, [\tau]_{\mathrm{S}_2})). \tag{52}$$

Since $\langle \Phi_{(N_1, N_2)}, \mathcal{H}(\Lambda) \Phi_{(N_1', N_2')} \rangle = 0$ and coefficients g are independent if $(N_1, N_2) \neq (N_1', N_2')$, we can find the lowest-energy state in each (N_1, N_2)-sector. Because the Hamiltonian $\mathcal{H}(\Lambda)$ has a rotational symmetry, all eigenstates for it have their representatives in the $M = 0$ or $\frac{1}{2}$ subspace according to the parity of N_e. Therefore our goal is reduced to find the state $\Phi_{(N_1, N_2)}$ which minimizes the expectation value $E_{(N_1, N_2)}$ defined by

$$E_{(N_1, N_2)} = \langle \Phi_{(N_1, N_2)}, \mathcal{H}(\Lambda) \Phi_{(N_1, N_2)} \rangle = \langle \Phi_{(N_1, N_2)}, \mathcal{H}^0_{\mathrm{hop}}(\Lambda) \Phi_{(N_1, N_2)} \rangle \tag{53}$$

where

$$\mathcal{H}^0_{\mathrm{hop}}(\Lambda) = \sum_{m=1,2} \mathcal{H}^{(t_m)}_{\mathrm{hop}}(\Lambda) + \sum_{m=1,2} \mathcal{H}^{(\tilde{v}_m)}_{\mathrm{hop}}(\Lambda) \tag{54}$$

within the $M = 0(\frac{1}{2})$ subspace, with the restriction $\langle \Phi_{(N_1,N_2)}, \Phi_{(N_1,N_2)} \rangle = 1$ and condition (50), and to find $S_{\rm tot}$ for it.

Hereafter we will consider the case that $N_{\rm e}$ is even. The case of odd $N_{\rm e}$ can be treated in the same way. We will work on the fixed (N_1, N_2)-sector, so we will abbreviate the subscript (N_1, N_2) if there is no confusion. We will put a bar on a state which is normalized and satisfies the finite-energy-state condition (50).

4.2. Proof for $\frac{1}{6} < \nu < \frac{1}{3}$

In this section we will prove that the Hamiltonian $\mathcal{H}(\Lambda)$ has the saturated ferromagnetic ground states for $N < N_{\rm e} < 2N(\frac{1}{6} < \nu < \frac{1}{3})$ in the limit that v_3 and U are infinitely large.

Now suppose

$$\overline{\Phi_{\rm G}} = \sum_{\gamma \in \Omega} g(\gamma)\Phi(\gamma) \tag{55}$$

gives the minimum value $E_{\rm G}$ of equation (53) within (N_1, N_2)-sector, where we have used the abbreviation, $\gamma = ({\rm S}_1, {\rm S}_2, ([\sigma]_{{\rm S}_1}, [\tau]_{{\rm S}_2}))$, and the set Ω is defined by

$$\Omega = \left\{ ({\rm S}_1, {\rm S}_2, ([\sigma]_{{\rm S}_1}, [\tau]_{{\rm S}_2})) \;\middle|\; \begin{array}{l} {\rm S}_1, {\rm S}_2 \subset {\rm L};\ |{\rm S}_1| = N_1,\ |{\rm S}_2| = N_2, \\ \sigma(x), \tau(x) = \uparrow, \downarrow;\ \sum_{x\in {\rm S}_1} \sigma(x) + \sum_{x \in {\rm S}_2} \tau(x) = 0 \end{array} \right\}. \tag{56}$$

We can easily see that $\langle \Phi(\gamma), \mathcal{H}^0_{\rm hop}(\Lambda)\Phi(\gamma')\rangle$ for $\gamma \neq \gamma'$ is 0, $-t_1$, or $-t_2$, i.e., is not positive. From this, we have

$$\begin{aligned} E_{\rm G} &= \sum_{\gamma,\gamma' \in \Omega} g(\gamma)g(\gamma')\langle \Phi(\gamma), \mathcal{H}^0_{\rm hop}(\Lambda)\Phi(\gamma')\rangle \\ &\geqslant \sum_{\gamma,\gamma' \in \Omega} |g(\gamma)||g(\gamma')|\langle \Phi(\gamma), \mathcal{H}^0_{\rm hop}(\Lambda)\Phi(\gamma')\rangle. \end{aligned} \tag{57}$$

It is obvious that if $\{g(\gamma)\}$ satisfies condition (50), then $\{|g(\gamma)|\}$ does as well. Therefore, the state

$$\overline{\Phi'_{\rm G}} = \sum_{\gamma \in \Omega} |g(\gamma)|\Phi(\gamma) \tag{58}$$

is also the lowest-energy state.

We will prove a positivity of expansion coefficients and the uniqueness of the lowest-energy state. Before we give a proof of these, we will introduce some notations. If $y+1 \notin {\rm S}_m$ and ${\rm S}'_m = \{y+1\} \cup {\rm S}_m \backslash \{y\}$, or $y-1 \notin {\rm S}_m$ and ${\rm S}'_m = \{y-1\} \cup {\rm S}_m \backslash \{y\}$, for one $y \in {\rm S}_m$, we write ${\rm S}'_m \leftrightarrow {\rm S}_m$. If $([\sigma']_{{\rm S}_1}, [\tau']_{{\rm S}_2})$ can be reduced to $([\sigma]_{{\rm S}_1}, [\tau]_{{\rm S}_2})$ by switching some pairs of $\sigma'(x)$ and $\tau'(x)$ for $x \in {\rm S}_1 \cap {\rm S}_2$ we also write $([\sigma']_{{\rm S}_1}, [\tau']_{{\rm S}_2}) \leftrightarrow ([\sigma]_{{\rm S}_1}, [\tau]_{{\rm S}_2})$. For γ' and γ, if one of the following relations is satisfied:

(i) ${\rm S}'_1 \leftrightarrow {\rm S}_1$, ${\rm S}'_2 = {\rm S}_2$, and $([\sigma]_{{\rm S}'_1}, [\tau]_{{\rm S}_2}) = ([\sigma]_{{\rm S}_1}, [\tau]_{{\rm S}_2})$
(ii) ${\rm S}'_2 \leftrightarrow {\rm S}_2$, ${\rm S}'_1 = {\rm S}_1$, and $([\sigma]_{{\rm S}_1}, [\tau]_{{\rm S}'_2}) = ([\sigma]_{{\rm S}_1}, [\tau]_{{\rm S}_2})$
(iii) $([\sigma']_{{\rm S}_1}, [\tau']_{{\rm S}_2}) \leftrightarrow ([\sigma]_{{\rm S}_1}, [\tau]_{{\rm S}_2})$

then we say they are directly connected and again write $\gamma' \leftrightarrow \gamma$. For γ_1 and γ_n, if we can find a sequence of $\{\gamma_l\}_{l=2}^{n-1}$ such that γ_l and γ_{l+1} are directly connected, then we say they are connected.

First we observe that any $g(\gamma)$ is non-zero. Now suppose that one $g(\gamma_1)$ with $\gamma_1 = ({\rm S}^1_1, {\rm S}^1_2, ([\sigma]_{{\rm S}^1_1}, [\tau]_{{\rm S}^1_2}))$ is zero. Then, from condition (50), we have $g(\gamma'_1) = 0$ for all γ'_1 such that $\gamma'_1 = ({\rm S}^1_1, {\rm S}^1_2, ([\sigma']_{{\rm S}^1_1}, [\tau']_{{\rm S}^1_2}))$ with $([\sigma']_{{\rm S}^1_1}, [\tau']_{{\rm S}^1_2}) \leftrightarrow ([\sigma]_{{\rm S}^1_1}, [\tau]_{{\rm S}^1_2})$. Since

$\langle\overline{\Phi}, (\mathcal{H}^0_{\rm hop}(\Lambda) - E_{\rm G})\overline{\Phi}\rangle$ is non-negative for any normalized finite-energy state $\overline{\Phi}$, by using the Schwartz inequality, we have

$$\langle\overline{\Phi}, (\mathcal{H}^0_{\rm hop}(\Lambda) - E_{\rm G})\overline{\Phi'_{\rm G}}\rangle|^2 \leqslant \langle\overline{\Phi}, (\mathcal{H}^0_{\rm hop}(\Lambda) - E_{\rm G})\overline{\Phi}\rangle\langle\overline{\Phi'_{\rm G}}, (\mathcal{H}^0_{\rm hop}(\Lambda) - E_{\rm G})\overline{\Phi'_{\rm G}}\rangle = 0. \quad (59)$$

This means

$$\langle\overline{\Phi}, \mathcal{H}^0_{\rm hop}(\Lambda)\overline{\Phi'_{\rm G}}\rangle = E_{\rm G}\langle\overline{\Phi}, \overline{\Phi'_{\rm G}}\rangle \quad (60)$$

holds for all finite-energy states. We choose $\overline{\Phi}$ as

$$\overline{\Phi} = \text{Const.} \sum_{([\sigma']_{S^1_1},[\tau']_{S^1_2})\leftrightarrow([\sigma]_{S^1_1},[\tau]_{S^1_2})} \Phi(S^1_1, S^1_2, ([\sigma']_{S^1_1}, [\tau']_{S^1_2})) \quad (61)$$

where Const. is a normalization factor, and the summation is over spin configurations $([\sigma']_{S^1_1}, [\tau']_{S^1_2})$ which satisfy the relation $([\sigma']_{S^1_1}, [\tau']_{S^1_2}) \leftrightarrow ([\sigma]_{S^1_1}, [\tau]_{S^1_2})$ including $([\sigma]_{S^1_1}, [\tau]_{S^1_2})$ itself. It is noted that this $\overline{\Phi}$ satisfies the finite-energy-state condition. From equation (60), noting $\langle\overline{\Phi}, \overline{\Phi'_{\rm G}}\rangle = 0$ by the assumption, we obtain

$$\sum_{([\sigma']_{S^1_1},[\tau']_{S^1_2})\leftrightarrow([\sigma]_{S^1_1},[\tau]_{S^1_2})} \sum_{\gamma\in\Omega} |g(\gamma)|\langle\Phi(S^1_1, S^1_2, ([\sigma']_{S^1_1}, [\tau']_{S^1_2})), \mathcal{H}^0_{\rm hop}(\Lambda)\Phi(\gamma)\rangle = 0. \quad (62)$$

For $\gamma' \neq \gamma$, $\langle\Phi(\gamma'), \mathcal{H}^0_{\rm hop}(\Lambda)\Phi(\gamma)\rangle$ is always non-positive, and therefore this equation holds only if $g(\gamma)$ is zero for all γ such that $\langle\Phi(S^1_1, S^1_2, ([\sigma']_{S^1_1}, [\tau']_{S^1_2})), \mathcal{H}^0_{\rm hop}(\Lambda)\Phi(\gamma)\rangle$ is non-zero. We can easily see that at least that is non-zero if γ is equal to $(S'_1, S'_2, ([\sigma]_{S'_1}, [\tau]_{S'_2}))$ with $S'_1 \leftrightarrow S^1_1$, $S'_2 = S^1_2$ and $([\sigma]_{S'_1}, [\tau]_{S'_2}) = ([\sigma]_{S^1_1}, [\tau]_{S^1_2})$, or $S'_1 = S^1_1$, $S'_2 \leftrightarrow S^1_2$ and $([\sigma]_{S'_1}, [\tau]_{S'_2}) = ([\sigma]_{S^1_1}, [\tau]_{S^1_2})$. Therefore, $g(\gamma)$ must be zero for such γ. From the above discussion, $\overline{\Phi'_{\rm G}} = 0$ or equivalently $\overline{\Phi_{\rm G}} = 0$, if any γ is connected to γ_1. As we can see below, for $N < N_{\rm e} < 2N$ any γ and γ' are actually connected. This leads to a contradiction, and in conclusion, $g(\gamma)$ cannot be zero.

We will prove the connectivity of γ in the case $N_1 \geqslant N_2$. A proof for the case $N_1 < N_2$ is the same. It is noted that $0 < N_2 < N$ because $N < N_{\rm e} < 2N$. Let $\gamma_0 = (S^0_1, S^0_2, ([\sigma^0]_{S^0_1}, [\tau^0]_{S^0_2}))$, where $S^0_m = \{1, \ldots, N_m\}$, $\sigma^0(x) = \downarrow$ for $1 \leqslant x \leqslant N_{\rm e}/2$, $\sigma^0(x) = \uparrow$ for $N_{\rm e}/2{+}1 \leqslant x \leqslant N_1$ and $\tau^0(y) = \uparrow$ for all $y \in S^0_2$. It is enough to show that any γ is connected to γ_0. Let us start with γ_1. We can easily see that it is (usually not directly) connected to $\gamma_2 = (S^0_1, S^0_2, ([\sigma]_{S^0_1}, [\tau]_{S^0_2}))$ where $([\sigma]_{S^0_1}, [\tau]_{S^0_2}) = ([\sigma]_{S^1_1}, [\tau]_{S^1_2})$. It is also obvious that γ_2 is directly connected to $\gamma_3 = (S^0_1, S^0_2, ([\sigma^3]_{S^0_1}, [\tau^3]_{S^0_2}))$, where $([\sigma^3]_{S^0_1}, [\tau^3]_{S^0_2})$ is obtained from $([\sigma]_{S^0_1}, [\tau]_{S^0_2})$ through the relations $\sigma^3(x) = \downarrow$ and $\tau^3(x) = \uparrow$ if $\sigma(x) = \uparrow$ and $\tau(x) = \downarrow$, and $\sigma^3(x) = \sigma(x)$ and $\tau^3(x) = \tau(x)$ if otherwise. Then there are three combinations of $\sigma^3(x)$ and $\tau^3(x)$, which are $(\sigma^3(x), \tau^3(x)) = (\uparrow, \uparrow)$, $(\downarrow, \downarrow)$, and $(\downarrow, \uparrow)$. Let us assume that the number of the combination $(\sigma^3(x), \tau^3(x)) = (\uparrow, \uparrow)$ is larger than that of $(\sigma^3(x), \tau^3(x)) = (\downarrow, \downarrow)$. The inverse case can be treated in the same way. Let us assume that $(\sigma^3(z), \tau^3(z)) = (\uparrow, \uparrow)$

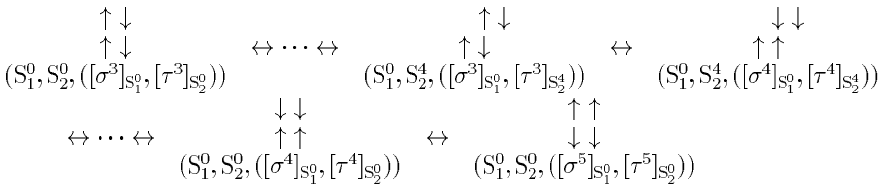

Figure 6. The connectivity between γ_3 and γ_5.

and $(\sigma^3(z+1), \tau^3(z+1)) = (\downarrow, \downarrow)$ for some $z \in \mathrm{L}$. In this case, we can find a sequence $(\mathrm{S}_1^0, \mathrm{S}_2^0, ([\sigma^3]_{\mathrm{S}_1^0}, [\tau^3]_{\mathrm{S}_2^0})) \leftrightarrow \cdots \leftrightarrow (\mathrm{S}_1^0, \mathrm{S}_2^4, ([\sigma^3]_{\mathrm{S}_1^0}, [\tau^3]_{\mathrm{S}_2^4})) \leftrightarrow (\mathrm{S}_1^0, \mathrm{S}_2^4, ([\sigma^4]_{\mathrm{S}_1^0}, [\tau^4]_{\mathrm{S}_2^4})) \leftrightarrow \cdots \leftrightarrow (\mathrm{S}_1^0, \mathrm{S}_2^0, ([\sigma^4]_{\mathrm{S}_1^0}, [\tau^4]_{\mathrm{S}_2^0})) \leftrightarrow (\mathrm{S}_1^0, \mathrm{S}_2^0, ([\sigma^5]_{\mathrm{S}_1^0}, [\tau^5]_{\mathrm{S}_2^0})) = \gamma_5$, where $\mathrm{S}_2^4 = \{1, \ldots, z-1, z+1, \ldots, N_2+1\}$, $(\sigma^5(z), \tau^5(z)) = (\downarrow, \uparrow)$, and $(\sigma^5(z+1), \tau^5(z+1)) = (\downarrow, \uparrow)$ (figure 6). Next, let us assume that $(\sigma^5(z), \tau^5(z)) = (\uparrow, \uparrow)$ and $(\sigma^5(z+1), \tau^5(z+1)) = (\downarrow, \uparrow)$ for some $z \in \mathrm{L}$. In this case, we can also find $\gamma_6 = (\mathrm{S}_1^0, \mathrm{S}_2^0, ([\sigma^6]_{\mathrm{S}_1^0}, [\tau^6]_{\mathrm{S}_2^0}))$ with $(\sigma^6(z), \tau^6(z)) = (\downarrow, \uparrow)$ and $(\sigma^6(z+1), \tau^6(z+1)) = (\uparrow, \uparrow)$, which is connected to γ_5 through a similar sequence. Therefore, we can find $\gamma' = (\mathrm{S}_1^0, \mathrm{S}_2^0, ([\sigma']_{\mathrm{S}_1^0}, [\tau']_{\mathrm{S}_2^0}))$ which is connected to γ_3 and whose spin configuration is such a one as $(\sigma'(x), \tau'(x)) = (\downarrow, \uparrow)$ for $1 \leqslant x \leqslant N'$ and $(\sigma'(x), \tau'(x)) = (\uparrow, \uparrow)$ for $N'+1 \leqslant x \leqslant N_2$ with some integer $N' \leqslant N_2$. We can easily see that γ' is connected to γ_0. Therefore, we can conclude that γ_1 is connected to γ_0.

From (57) we have

$$\sum_{\gamma,\gamma' \in \Omega} (g(\gamma)g(\gamma') - |g(\gamma)||g(\gamma')|)\langle \Phi(\gamma), \mathcal{H}_{\mathrm{hop}}^0(\Lambda)\Phi(\gamma')\rangle = 0. \tag{63}$$

Since $\langle \Phi(\gamma), \mathcal{H}_{\mathrm{hop}}^0(\Lambda)\Phi(\gamma')\rangle$ and $(g(\gamma)g(\gamma') - |g(\gamma)||g(\gamma')|)$ are non-positive, and any $g(\gamma)$ is non-zero, by using the connectivity of γ again, we find that all $g(\gamma)$ have the same sign. We choose the plus sign. From this, we can see the uniqueness of the lowest-energy state. Suppose that there are two linearly independent lowest-energy states. Without loss of generality we can assume that these are orthogonal. Then, any linear combination of these states, which might have a negative coefficient, is also the lowest-energy state, and this leads to a contradiction. Therefore, we can conclude that the lowest-energy state is unique.

Now we can easily show that the lowest-energy state in each (N_1, N_2)-sector, has $S_{\mathrm{tot}} = N_{\mathrm{e}}/2$ and therefore so does the ground state. The lowest-energy state is unique, so it must be the eigenstate for $(\boldsymbol{S}_{\mathrm{tot}})^2$. We will prepare the state $\Phi_{\mathrm{ferro}} = \prod_{x \in \mathrm{S}_1^0} a_{[x,1],\uparrow}^\dagger \prod_{x \in \mathrm{S}_2^0} a_{[x,2],\uparrow}^\dagger \Phi_0$, whose S_{tot} is $N_{\mathrm{e}}/2$. The positivity of expansion coefficients for $\overline{\Phi_{\mathrm{G}}}$ implies that $\langle (S_{\mathrm{tot}}^-)^{N_{\mathrm{e}}/2}\Phi_{\mathrm{ferro}}, \overline{\Phi_{\mathrm{G}}}\rangle$ is non-zero, where S_{tot}^-, the total spin lowering operator, is defined by $S_{\mathrm{tot}}^- = \sum_{j \in \Lambda}(S_j^{(1)} - iS_j^{(2)})$. Therefore, we can conclude that S_{tot} for $\overline{\Phi_{\mathrm{G}}}$ is $N_{\mathrm{e}}/2$.

4.3. Extension to the electron filling factors $\nu_0 < \nu < \frac{1}{3}$ with $0 \leqslant \nu_0 < \frac{1}{6}$

We can extend the result in the previous section to $\nu_0 < \nu < \frac{1}{3}$ with $0 \leqslant \nu_0 < \frac{1}{6}$ if the lower two bands have an overlap in the sense that $(\varepsilon_1^{(1)}, \varepsilon_N^{(1)}] \cap (\varepsilon_1^{(2)}, \varepsilon_N^{(2)}] \neq \emptyset$.

First we observe that in the case of $v_3, U \to \infty$ and $(\varepsilon_1^{(1)}, \varepsilon_N^{(1)}] \cap (\varepsilon_1^{(2)}, \varepsilon_N^{(2)}] \neq \emptyset$, there exists ν_0 with $0 \leqslant \nu_0 < \frac{1}{6}$ such that for $\nu \leqslant \nu_0$ the ground states of $\mathcal{H}(\Lambda)$ are in the $(N_{\mathrm{e}}, 0)$- or $(0, N_{\mathrm{e}})$-sector, but for $\nu_0 < \nu$ they are in the (N_1, N_2)-sector with non-zero N_1 and N_2. To prove the above statement we will consider the Hamiltonian $\mathcal{H}(\Lambda) + \mathcal{H}_J(\Lambda)$, where $\mathcal{H}_J(\Lambda)$ is defined by

$$\mathcal{H}_J(\Lambda) = J \sum_{x \in \mathrm{L}\setminus\{N\}} \sum_{m=1,2} \left(\frac{\tilde{n}_{[x,m]}\tilde{n}_{[x+1,m]}}{4} - \tilde{\boldsymbol{S}}_{[x,m]} \cdot \tilde{\boldsymbol{S}}_{[x+1,m]} \right). \tag{64}$$

Since $\langle \Phi_{(N_1,N_2)}, \mathcal{H}_J(\Lambda)\Phi_{(N_1',N_2')}\rangle = 0$ if $(N_1, N_2) \neq (N_1', N_2')$, we can again work on the fixed (N_1, N_2)-sector in the limit $v_3, U \to \infty$. By following the same method as in the previous section, we can prove that the lowest-energy state of the Hamiltonian $\mathcal{H}(\Lambda) + \mathcal{H}_J(\Lambda)$ within each (N_1, N_2)-sector has $S_{\mathrm{tot}} = N_{\mathrm{e}}/2$ and is unique up to its trivial degeneracy due to the rotational symmetry for the electron filling factor $0 < \nu \leqslant \frac{1}{3}$ if J is positive. By the continuity of the energy, it is concluded that the lowest energy of $\mathcal{H}(\Lambda)$ within each (N_1, N_2)-sector is $\sum_{m=1}^2 \sum_{l=1}^{N_m} \varepsilon_l^{(m)}$ although the uniqueness may or may not hold. For a given N_{e}, we will find

the pair of (N_1, N_2) which minimizes $\sum_{m=1}^{2}\sum_{l=1}^{N_m}\varepsilon_l^{(m)}$. Therefore, according to the values of t, v_1 and v_2, we find ν_0 with $0 \leqslant \nu_0 < \frac{1}{6}$ such that for ν satisfying $\nu \leqslant \nu_0$ the minimum value of $\sum_{m=1}^{2}\sum_{l=1}^{N_m}\varepsilon_l^{(m)}$ is given by $(N_e, 0)$ or $(0, N_e)$, but for ν satisfying $\nu_0 < \nu$ it is given by (N_1, N_2) with non-zero N_1 and N_2.

Now suppose that we find that the ground states of $\mathcal{H}(\Lambda)$ are within a (N_1, N_2)-sector with $0 < N_1 < N$ and $0 < N_2 < N$. By following the same method as in the previous section again, we can reach the conclusion that the lowest-energy state within the (N_1, N_2)-sector has $S_{tot} = N_e/2$ and is unique up to its trivial degeneracy. Therefore $\mathcal{H}(\Lambda)$ in the limit $v_3, U \to \infty$ has the saturated ferromagnetic ground states for $\nu_0 < \nu < \frac{1}{3}$, where ν_0 takes a value in the range $0 \leqslant \nu_0 < \frac{1}{6}$ depending on the values of t, v_1 and v_2.

We end the proof by giving two remarks.

If we impose periodic boundary conditions, we will meet a sign problem caused by exchange of fermion operators. We can see, at least, that the lowest-energy state has $S_{tot} = N_e/2$ in the (N_1, N_2)-sector with both N_1 and N_2 being odd integers, but cannot see anything in other sectors.

We can construct models in higher dimensions similar to the present model. However, it is almost impossible to extend the argument in this section to models in higher dimensions, where we cannot avoid a sign problem of fermion operators. It is an interesting problem to investigate magnetic properties of the effective Hamiltonian in the limit of the infinitly large band gap in higher dimensions.

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